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A characterization of quasi-Schwartz linear mappings

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ABSTRACT

The notion of quasi-Schwartz mappings, introduced by Randtke, is a natural generalization of the concept of precompact linear mappings. It is shown that a linear mapping T from a locally convex space E into another F is a quasi-Schwartz mapping if and only if there exist a o-neighbourhood V in E and an absolutely convex bounded subset B of F such that $T(V)$ is precompact in the normed space $(\bigcup_{n\geq 1} n\, B, r_B)$. As an application, we obtain Schauder's theorem on the duality properties of precompact and compact mappings. Another application is to establish a factorization theorem for quasi-Schwartz mappings; it is shown that for every quasi-Schwartz mapping $T: E\to F$ there exist a vector subspace H of c_0 and quasi-Schwartz mappings $T_1: E\to H$ and $T_2: H\to F$ such that $T=T_2 \circ T_1$. This is a generalization of the factorization theorem of precompact mappings, due to Terzioglu and Randtke.

1. INTRODUCTION

Throughout this paper E and F will denote Hausdorff locally convex spaces. If B is an absolutely convex bounded subset of F, then we denote by $F(B)$ the normed vector space obtained by furnishing the vector subspace $\bigcup_{n\geq 1} n\, B$ of F generated by B equipped with the norm r_B induced by B. A linear mapping $T: E\to F$ is called a *quasi-Schwartz linear mapping* [2] (*Precompact-bounded linear mapping* in the terminology of [5]) if there exists a precompact seminorm p on E (for definition, see the next section) such that $\{Tx : p(x)\leq 1\}$ is a bounded subset of F.

In this paper we show that a linear mapping $T: E\to F$ is quasi-Schwartz if and only if there exist an absolutely convex o-neighbourhood V in E and an absolutely convex bounded subset B of F such that $T(V)$ is a precompact subset of the normed space $F(B)$ (Theorem 2), and this is the case if and only if its adjoint mapping T' is a quasi-Schwartz linear mapping from $(F', \beta(F', F))$ into $(E', \beta(E', E))$. As an application, we obtain Schauder's theorem on the duality properties of precompact and compact mappings. Another application is to establish a factorization theorem for quasi-Schwartz linear mappings (see Theorem 5), this is a generalization of the factorization theorem of precompact mappings, due to Terzioglu [4] and Randtke [3] (see Köthe [1, § 42, 8 (4)]).

2. MAIN RESULTS

A subset K of F is said to be *b-precompact* if there exists an absolutely convex bounded subset B of F such that K is precompact in the normed space $F(B)$. As the norm-topology on $F(B)$ is finer than the relative topology on $F(B)$ induced by the original topology, it follows that b-precompact subsets of F are precompact; but the converse is, in general, not true. Furthermore, if F is metrizable then it is well-known (see Köthe [1, § 42, 1 (13)]) that every precompact subset of F is b-precompact.

Let us say temporarily that a linear mapping $T: E \to F$ is *b-precompact* if there exists a o-neighbourhood V in E such that $T(V)$ is a b-precompact subset of F. Clearly every b-precompact linear mapping from E into F is precompact (namely, it sends some o-neighbourhood in E into a precompact subset of F), but the converse is, in general, not true.

In order to verify one of our main results we require the following result.

LEMMA 1. *Let E, F, G be locally convex spaces, let $T: E \to F$ and $S: F \to G$ be continuous linear mappings. If one of T and S is b-precompact, then $S \circ T: E \to G$ is a b-precompact linear mapping.*

PROOF. (i) Suppose that T is b-precompact. Then there exist a o-neighbourhood V in E and an absolutely convex bounded subset B of F such that $T(V)$ is a precompact subset of the normed space $F(B)$. The continuity of S ensures that $C = S(B)$ is an absolutely convex bounded subset of G. Furthermore, the restriction $S|_B$ on $F(B)$ of S is a continuous linear mapping from the normed space $F(B)$ into another $G(C)$, hence $S|_B(T(V)) = S \circ T(V)$ is a precompact subset of the normed space $G(C)$. Therefore $S \circ T$ is a b-precompact linear mapping.

(ii) Suppose that S is b-precompact. Then there exists a o-neighbourhood U in F such that $S(U)$ is a b-precompact subset of G. The continuity of T ensures that there exists a o-neighbourhood V in E such that $T(V) \subset U$, hence $(S \circ T)V \subset S(U)$ and thus $(S \circ T)V$ is a b-precompact subset of G. This shows that $S \circ T$ is a b-precompact linear mapping.

Randtke [2] calls a seminorm p on E *precompact* if there exist an $[\zeta_n] \in c_0$ (the Banach space of zero-convergent sequences) and an equicontinuous sequence $\{u_n'\}$ in E' such that

$$p(x) \leq \sup_n \{ |\zeta_n \langle x, u_n' \rangle| \} \text{ for all } x \in E.$$

A linear mapping $T:E\to F$ is called a *quasi-Schwartz linear mapping* if there exists a precompact seminorm p on E such that $\{Tx:p(x)\leq 1\}$ is a bounded subset of F.

It is easily seen that a linear mapping $T:E\to F$ is quasi-Schwartz if and only if T is the composition of the following continuous linear mappings

$$E\xrightarrow{Q}X\xrightarrow{\tilde{T}}Y\xrightarrow{J}F,$$

where X and Y are normed spaces and $\tilde{T}:X\to Y$ is a precompact mapping (see [2, (2.11)] or [5, (1.3.4)]). As each precompact subset of Y is b-precompact, it follows that $\tilde{T}:X\to Y$ is a b-precompact mapping, and hence from Lemma 1 that T is a b-precompact mapping. The converse is also valid as shown by the following result.

THEOREM 2. *A linear mapping $T:E\to F$ is quasi-Schwartz if and only if T is a b-precompact linear map.*

PROOF. It has only to prove the sufficiency. Let V be a o-neighbourhood, and let B be an absolutely convex bounded subset B of F such that $T(V)$ is a precompact subset of the normed space $F(B)$. Then T is a precompact mapping from E into the normed space $F(B)$. In view of [2, (2.10)] there exist an $[\zeta_n]\in c_0$ and an equicontinuous sequence $\{u_n'\}$ in E' such that

$$r_B(Tx)\leq \sup_n \{|\zeta_n\langle x,u_n'\rangle|\} \text{ for all } x\in E.$$

The seminorm p on E, defined by

$$p(x)=\sup\{|\zeta_n\langle x,u_n'\rangle|\} \text{ for all } x\in E,$$

in a precompact seminorm such that $\{Tx:p(x)\leq 1\}\subseteq B$, hence T is quasi-Schwartz linear mapping.

COROLLARY 3. (Randtke [2, (2.9) and (2.12)]). *For a given locally convex space F, the following statements are equivalent.*

(a) *Every precompact subset of F is b-precompact.*

(b) *For any locally convex space E, any precompact linear mapping from E into F is a quasi-Schwartz linear mapping.*

In particular, if F is metrizable, then every precompact linear mapping from E into F is quasi-Schwartz.

PROOF. The implication (a)$\Rightarrow$(b) follows from Theorem 2. To prove the implication (b)$\Rightarrow$(a), let B be any precompact, absolutely convex subset of F. Then the embedding mapping $j_B:F(B)\to F$ is a precompact linear mapping, hence B is b-precompact by Theorem 2.

Finally, since every metrizable space has the property mentioned in (a), the conclusion follows.

COROLLARY 4. *Let $T:E\to F$ be a weakly continuous linear mapping. Then T is quasi-Schwartz if and only if its adjoint mapping T' is a quasi-Schwartz linear mapping from $(F',\beta(F',F))$ into $(E',\beta(E',E))$, where $\beta(F',F)$ and $\beta(E',E)$ are the strong topologies, provided that E is infrabarrelled.*

PROOF. Follows from Theorem 2 and Köthe [1, § 42, 1 (10)].

Combining Corollary 3, we see that the preceding result is a generalization of Schauder's theorem on the duality properties of precompact and compact mappings (see Köthe [1, § 42, 1 (7)]).

Terzioglu [4] and Randtke [3] show that a linear mapping T from a normed space X into another Y is precompact if and only if it has a precompact factorization through a vector subspace of c_0 (see Köthe [1, § 42, 8 (4)]). In fact, this result is still true whenever X is a locally convex space and Y is a metrizable locally convex space (see [5, (1.1.7)]). In view of Theorem 2, we are able to give a factorization theorem for quasi-Schwartz linear mappings as follows.

THEOREM 5. *Every quasi-Schwartz linear mapping $T:E\to F$ has a quasi-Schwartz factorization through a vector subspace H of c_0, namely there exist a vector subspace H of c_0 and quasi-Schwartz linear mappings*

$$E\xrightarrow{T_1}H\xrightarrow{T_2}F$$

such that $T=T_2\circ T_1$.

PROOF. In view of Theorem 2, there exist a o-neighbourhood V in E and an absolutely convex bounded subset B of F such that $T(V)$ is a precompact subset of the normed space $F(B)$, hence T is precompact as a mapping from E into $F(B)$. We use the argument of [3] to verify that the precompact linear mapping $T:E\to F(B)$ has a precompact factorization through a vector subspace of c_0.

In fact, it is known from [2, (2.10)] (or [5, (1.1.6)]) that there exist an $[\zeta_n]\in c_0$ and an equicontinuous sequence $\{u_n'\}$ in E' such that

$$(1)\qquad r_B(Tx)\leq \sup_n \{|\zeta_n^2\langle x,u_n'\rangle|\} \text{ for all } x\in E.$$

We define a mapping $T_1:E\to c_0$ by

$$T_1x=[\zeta_n\langle x,u_n'\rangle] \text{ for all } x\in E,$$

and let $H=T_1E$. Then T_1 is a precompact linear mapping from E into a vector subspace H of c_0. The diagonal transformation $D:c_0\to c_0$, defined by

$$D([\eta_n])=[\zeta_n\eta_n] \text{ for all } [\eta_n]\in c_0,$$

is a compact linear mapping such that $\operatorname{Ker} D\circ T_1\subset\operatorname{Ker} T$ since r_B is a norm on $F(B)$ on account of (1). Therefore there exists a continuous linear map S from the vector subspace $D(H)$ of c_0 into $F(B)$ such that $T=S\circ D\circ T_1$, consequently $S\circ D$ is a precompact linear map from H into $F(B)$. This obtains our assertion.

As H and $F(B)$ are normed spaces, it follows that $T_1: E \to H$ and $S \circ D$: $H \to F(B)$ are quasi-Schwartz linear mappings. If $j_B: F(B) \to F$ is the embedding mapping, then $T_2 = j_B \circ S \circ D$ is a quasi-Schwartz linear mapping from a vector subspace H of c_0 into F. Clearly

$$T = S \circ D \circ T_1 = (j_B \circ S \circ D) \circ T_1 = T_2 \circ T_1,$$

thus T is the composition of two quasi-Schwartz linear mappings.

Of course, the converse of the preceding result holds on account of Lemma 1.

In view of Corollary 3, the preceding result is a generalization of the factorization theorem of Terzioglu [4] and Randtke [3] (see Köthe [1, § 42, 8 (4)]).

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